

# A New Route to the Interpretation of Hopf Invariant

REN Ji-rong, LI Ran,\* and DUAN Yi-shi

*Institute of Theoretical Physics, Lanzhou University, Lanzhou, 730000, China*

## Abstract

We discuss an object from algebraic topology, Hopf invariant, and reinterpret it in terms of the  $\phi$ -mapping topological current theory. The main purpose of this paper is to present a new theoretical framework which can directly give the relationship between Hopf invariant and the linking numbers of the higher dimensional submanifolds of Euclidean space  $R^{2n-1}$ . For the sake of this purpose we introduce a topological tensor current which can naturally deduce the  $(n-1)$  dimensional topological defect in  $R^{2n-1}$  space. If these  $(n-1)$  dimensional topological defects are closed oriented submanifolds of  $R^{2n-1}$ , they are just the  $(n-1)$  dimensional knots. The linking number of these knots is well defined. Using the inner structure of the topological tensor current, the relationship between Hopf invariant and the linking numbers of the higher dimensional knots can be constructed.

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\*Electronic mail: liran05@lzu.cn

## I. INTRODUCTION

Results from the pure mathematical literature are of little difficulty for physicists and can be made accessible to physicists by introducing them in common physical methods. In this paper, we will discuss an object from algebraic topology, the Hopf invariant, and reinterpret it in terms of the  $\phi$ -mapping topological current theory proposed by Duan. Hopf studied the third homotopy group of the 2-sphere and showed that this group is non-trivial; later Hopf invented more non-trivial fibration and finally obtained a series of invariants that now bear his name. A review article is appeared in Ref.[1] in which the author pointed out the Hopf fibration occurs in at least seven different situation in theoretical physics in various guises.

It is well known that Hopf invariant is characterized by the homotopy group  $\pi_{2n-1}(S^n)$ . In the case that  $n = 2$ , the Hopf invariant  $\pi_3(S^2)$  as the linking numbers of knots has many applications in condensed matter physics and gauge field theory. In fact, we can take the Hopf map  $S^3 \rightarrow S^2$  as a projection in the sense that  $S^3$  is a principle fibre bundle over the base space  $S^2$  with the structure group  $U(1)$ . The standard fibre of Hopf bundle is  $S^1$  which is the inverse image of the point of  $S^2$  under the Hopf map. In 3-sphere,  $S^1$  is homeomorphous with knot and the topological quantity to describe the topological characteristics of these knots(standard fibres) is Hopf invariant. As applications, for example, a version of nonlinear  $O(3)$   $\sigma$ -model introduced by Faddeev[2] allows formation of the stable finite-length solitons which may have a form of knot or vortex loop. The topological quantum number of these stable defects is just described by the Hopf invariant  $\pi_3(S^2)$  which is the linking numbers of these knots or vortex loops. In gauge field theory, Chern-Simons action have a deep relationship with Hopf invariant. In [3], by virtue of the decomposition of  $U(1)$  gauge potential and  $\phi$ -mapping topological current theory, one of authors have pointed out that  $U(1)$  Chern-Simons action is the total sum of all the self-linking and linking numbers of the knots inherent in Chern-Simons field theory.

We [4] have studied the Hopf invariant  $\pi_3(S^2)$  by using the  $\phi$ -mapping method. Through the spinor representation of Hopf map, the inner topological structure is discovered which indicates that Hopf invariant is just the winding number  $\pi_3(S^3)$ . We also obtain a precise expression for Hopf invariant from which one can relate the Hopf invariant with the linking number and self-linking numbers of Knots. In this paper, almost using the same method,

we will study the general Hopf invariant  $\pi_{2n-1}(S^n)$  and give a precise expression for it. It is revealed that Hopf invariant is the linking numbers of the higher dimensional knots or higher dimensional closed oriented submanifolds of  $R^{2n-1}$ . The route to discover the interpretation of Hopf invariant appeared in this paper is new, at least as far as we know. This will increase our knowledge about Hopf invariant without doubt. We have mainly employed the tensor and coordinate language which may be nervous for some readers. But the result is really perfect. Firstly, we will review Hopf invariant according to the general textbook.

Hopf map[5] is the differentiable map between the spheres of different dimensions

$$f : S^{2n-1} \rightarrow S^n. \quad (1)$$

Consider a mapping

$$n : R^{2n-1} \rightarrow S^n, \quad (2)$$

which gives a  $(n+1)$  dimensional unit vector  $n^A (A = 1, 2, \dots, n+1)$ . We add the boundary condition

$$n(x)|_{x \rightarrow \infty} \rightarrow n_0, \quad (3)$$

where  $n_0$  is fixed vector or a fixed point on  $S^n$ . It is to say that we assume infinite distance points of  $R^{2n-1}$  is mapped to the same point on  $S^n$ , and therefore the spatial infinity can be efficiently contracted to a point, i. e. ,  $R^{2n-1} \approx S^{2n-1}$ . Thus, the unit vector  $n(x)$  provides us the mapping  $S^{2n-1} \rightarrow S^n$  which is just the Hopf map. Under the boundary condition (3), We will not distinguish  $R^{2n-1}$  with  $S^{2n-1}$ . So it is intuitively clear to choose Euclidean metric in the following discussion.

In terms of  $(n+1)$  dimensional unit vector  $n^A$ , the unit volume element of  $S^n$  can be constructed like this[6]

$$\tau = \frac{1}{A(S^n)n!} \epsilon_{A_1 A_2 \dots A_{n+1}} n^{A_1} dn^{A_2} \wedge \dots \wedge dn^{A_{n+1}}, \quad (4)$$

which is a closed  $n$ -form satisfying

$$\int_{S^n} \tau = 1. \quad (5)$$

Pulling the unit volume element  $\tau$  of  $S^n$  back to  $R^{2n-1}$ ,  $n^* \tau$  must be an exact form. One have

$$d\omega = n^* \tau, \quad (6)$$

where  $\omega$  is a  $(n-1)$ -form on  $R^{2n-1}$ . Hopf invariant[7] which depends only on the homotopy group  $\pi_{2n-1}(S^n)$  can be written in the form of integral

$$H = \int_{S^{2n-1}} \omega \wedge d\omega . \quad (7)$$

There are three properties for Hopf invariant[8]. The first, Hopf invariant is independent of the choice of  $(n-1)$ -form  $\omega$ . This gives us a great convenience in the discussion. The second, for odd  $n$  the Hopf invariant is 0. In this paper, we only consider the case that  $n$  is even. The third, homotopic maps have the same Hopf invariant. Actually, the Hopf invariant is always interpreted as the linking number of the preimages  $N_k = f^{-1}(p)$  and  $N_l = f^{-1}(q)$  of any two different values of Hopf mapping  $f$ .

Although until now the general Hopf invariant have not found as many applications in physics as the Hopf invariant  $\pi_3(S^2)$ , one must notice in this paper that these higher dimensional closed oriented submanifolds are deduced naturally from the topological tensor current and their generation mechanism is very similar with the topological  $p$ -branes[9]. In string theory, an important aspect is the extra dimension theory which implies that spacetime is fundamentally higher dimensional with  $(4+n)$  spacetime dimensions. Except the usual four dimensional Einstein space-time manifold, there exists extra  $n$  dimension space which usually shrinks in a very small distance. We think that Hopf invariant may be a suitable quantity to describe the distorting and linking of the extra dimensional space.

The main purpose in this paper is to present a new theoretical framework which directly gives the relationship between Hopf invariant and the linking numbers of the higher dimensional knots. For this purpose we introduce a topological tensor current which can naturally deduce the  $(n-1)$  dimensional topological defect in  $R^{2n-1}$  space. If these  $(n-1)$  dimensional topological defects are closed oriented submanifolds of  $R^{2n-1}$ , they are just the  $(n-1)$  dimensional knots whose linking number is well defined. Using the inner structure of the topological tensor current, the relationship between Hopf invariant and the linking numbers of the higher dimensional knots can be constructed.

## II. INNER STRUCTURE OF TOPOLOGICAL CURRENT

Recall that the pulling back of  $\tau$  is an exact  $n$ -form and Hopf invariant is independent of the choice of  $(n-1)$ -form  $\omega$ , so one can construct  $(n-1)$ -form  $\omega$  like this in terms of the  $n$

dimensional unit vector  $m^a(x)$  ( $a = 1, 2, \dots, n$ )

$$\omega = \frac{1}{A(S^{n-1})(n-1)!} \epsilon_{a_1 a_2 \dots a_n} m^{a_1} dm^{a_2} \wedge \dots \wedge dm^{a_n}. \quad (8)$$

The unit vector field  $m^a(x)$  is the function of  $R^{2n-1}$ , where  $x$  is the coordinates of  $R^{2n-1}$ . In fact, noticing that  $n$  is even, this expression for  $\omega$  which shows the relationship of the Gauss mappings between the  $n$ -dimension and the  $(n+1)$ -dimension can be directly deduced from Eq.(4) and (6) in terms of Gauss-Bonnet-Chern theorem[10] with the help of  $(n+1)$ -dimensional Euclidean space  $R^{n+1}$ .

According to  $\phi$ -mapping topological current theory[11],  $d\omega$  is of the form of  $(n-1)$  order topological current. We introduce a topological tensor current

$$\begin{aligned} j^{\mu_1 \mu_2 \dots \mu_{n-1}} &= \frac{1}{(n-1)!} \epsilon^{\mu_1 \mu_2 \dots \mu_{2n-1}} \partial_{\mu_n} \omega_{\mu_{n+1} \mu_{n+2} \dots \mu_{2n-1}} \\ &= \frac{1}{A(S^{n-1})(n-1)!} \epsilon^{\mu_1 \mu_2 \dots \mu_{2n-1}} \epsilon_{a_1 a_2 \dots a_n} \partial_{\mu_n} m^{a_1} \dots \partial_{\mu_{2n-1}} m^{a_n}. \end{aligned} \quad (9)$$

Obviously the topological tensor current is identically conserved,

$$\partial_{\mu_i} j^{\mu_1 \mu_2 \dots \mu_{n-1}} = 0, \quad i = 1, 2, \dots, n-1. \quad (10)$$

In the  $\phi$ -mapping theory, the unit vector  $m^a(x)$  should be further determined by the smooth vectors  $\phi^a(x)$ , i.e.

$$m^a(x) = \frac{\phi^a(x)}{\|\phi\|}, \quad \|\phi\| = \phi^a \phi^a. \quad (11)$$

In our theory the smooth vector field  $\phi^a(x)$  may be looked upon as the generalized order parameters for topological defects which is corresponding to the zero points of  $\phi^a(x)$ .

Substituting Eq.(11) into Eq.(9) and noticing

$$dm^a(x) = \frac{1}{\|\phi\|} d\phi^a(x) + \phi^a(x) d\frac{1}{\|\phi\|}, \quad (12)$$

we have

$$j^{\mu_1 \mu_2 \dots \mu_{n-1}} = \frac{1}{A(S^{n-1})(n-1)!} \epsilon^{\mu_1 \mu_2 \dots \mu_{2n-1}} \epsilon_{a_1 a_2 \dots a_n} \partial_{\mu_n} \left( \frac{\phi^{a_1}}{\|\phi\|} \right) \partial_{\mu_{n+1}} \phi^{a_2} \dots \partial_{\mu_{2n-1}} \phi^{a_n}. \quad (13)$$

Defining the rank- $(n-1)$  Jacobian tensor  $D^{\mu_1 \mu_2 \dots \mu_{n-1}} \left( \frac{\phi}{x} \right)$  of  $\phi$  as

$$\epsilon^{\mu_1 \mu_2 \dots \mu_{2n-1}} \partial_{\mu_1} \phi^{a_1} \dots \partial_{\mu_{n-1}} \phi^{a_n} = \epsilon^{a_1 a_2 \dots a_n} D^{\mu_1 \mu_2 \dots \mu_{n-1}} \left( \frac{\phi}{x} \right) \quad (14)$$

and noticing

$$\epsilon_{a_1 a_2 \dots a_n} \epsilon^{a a_2 \dots a_n} = \delta_{a_1}^a (n-1)!, \quad (15)$$

it follows that

$$j^{\mu_1 \mu_2 \dots \mu_{n-1}} = \frac{1}{A(S^{n-1})} \frac{\partial}{\partial \phi^a} \left( \frac{\phi^a}{\|\phi\|^n} \right) D^{\mu_1 \mu_2 \dots \mu_{n-1}} \left( \frac{\phi}{x} \right). \quad (16)$$

Using the relationship

$$\frac{\phi^a}{\|\phi\|^n} = \begin{cases} -\frac{1}{(n-2)} \frac{\partial}{\partial \phi^a} \left( \frac{1}{\|\phi\|^{n-2}} \right) & \text{for } n > 2, \\ \frac{\partial}{\partial \phi^a} \ln \|\phi\| & \text{for } n = 2, \end{cases} \quad (17)$$

and the Green function formula in  $\phi$ -space

$$\Delta_\phi \left( \frac{1}{\|\phi\|^{n-2}} \right) = -(n-2) A(S^{n-1}) \delta^{(n)}(\phi), \quad (18)$$

$$\Delta_\phi (\ln \|\phi\|) = 2\pi \delta^{(2)}(\phi), \quad (19)$$

where  $\Delta_\phi = \frac{\partial^2}{\partial \phi^a \partial \phi^a}$  is the  $d$ -dimensional Laplacian operator in  $\phi$  space, we obtain a  $\delta$ -function like topological tensor current

$$j^{\mu_1 \mu_2 \dots \mu_{n-1}} = \delta^{(n)}(\phi) D^{\mu_1 \mu_2 \dots \mu_{n-1}} \left( \frac{\phi}{x} \right), \quad (20)$$

and find that  $j_{\mu_1 \mu_2 \dots \mu_{n-1}} \neq 0$  only when  $\phi = 0$ . So, it is essential to discuss the solutions of the equations

$$\phi^a(x) = 0, \quad a = 1, \dots, n. \quad (21)$$

The solution of the above equation plays an crucial role in realization of the  $\tilde{p}$ -brane scenario[9]. Suppose that the vector field  $\phi(x)$  possesses  $l$  isolated zeroes, according to the implicit function theorem[12], when the zeroes are regular points of  $\phi$ -mapping, i.e., the rank of the Jacobian matrix  $[\partial_\mu \phi^a]$  is  $n$ , the solution of equation

$$\begin{cases} \phi^1(x^1, x^2, \dots, x^{2n-1}) = 0, \\ \dots, \\ \phi^n(x^1, x^2, \dots, x^{2n-1}) = 0. \end{cases} \quad (22)$$

can be parameterized as

$$x^\mu = x_i^\mu(u^1, u^2, \dots, u^{n-1}), \quad (23)$$

where  $i$  denotes the  $i$ th solution. Eq.(23) denotes that the  $(n-1)$ -dimensional submanifold  $N_k$  embeds in  $R^{2n-1}$  space with  $u^I (I = 1, 2, \dots, n-1)$  being the local ordinates of

submanifold  $N_k$ . We assume that  $N_k(k = 1, 2, \dots, l)$  are knotted(closed smooth oriented) submanifolds of  $R^{2n-1}$ . It will be seen in the following that Hopf invariant is just the linking numbers of these knotted submanifolds. For this purpose, we will study the inner structure of the topological current, i.e. expand the  $n$ -dimensional  $\delta$ -function on the  $(n-1)$ -dimensional singular submanifold  $N_k$ .

From the above discussions, we see that the topological current will not disappear only when  $\phi = 0$ . Here we will focus on the zero points of order parameter field  $\phi$  and will search for the inner topological structure of the topological current. It can be proved that there exists a  $n$ -dimensional submanifold  $M_k$  in  $R^{2n-1}$  with local parametric equation

$$x^\mu = x_i^\mu(v^1, v^2, \dots, v^n), \quad (24)$$

which is transversal to every  $N_i$  at the point  $p_i$  with metric

$$g_{\mu\nu} B_I^\mu B_a^\nu|_{p_i} = 0, \quad I = 1, \dots, n-1, \quad a = 1, \dots, n, \quad (25)$$

where

$$\frac{\partial x^\mu}{\partial u^I} = B_I^\mu, \quad \frac{\partial x^\nu}{\partial v^a} = B_a^\nu, \quad \mu, \nu = 1, 2, \dots, 2n-1, \quad (26)$$

are tangent vectors of  $N_i$  and  $M_i$  respectively and  $g_{\mu\nu} = \delta_{\mu\nu}$  for Euclidean space  $R^{2n-1}$ . As we have pointed out in Ref.[13], the unit vector field defined as  $m : \partial M_i \rightarrow S^{n-1}$  gives a Gauss map, and the generalized Winding Number can be given by the map

$$W_i = \frac{1}{A(S^{n-1})(n-1)!} \int_{\partial M_i} m^*(\varepsilon_{A_1 \dots A_n} m^{A_1} dm^{A_2} \wedge \dots \wedge dm^{A_n}), \quad (27)$$

where  $m^*$  denotes the pull back of map  $m$  and  $\partial M_i$  is the boundary of the neighborhood  $M_i$  of  $p_i$  on  $R^{2n-1}$  with  $p_i \notin \partial M_i$ ,  $M_i \cap M_j = \emptyset$ . It means that, when the point  $v^b$  covers  $\partial M_i$  once, the unit vector  $m$  will cover the unit sphere  $S^{n-1}$  for  $W_i$  times. Using the Stokes' theorem in exterior differential form and duplicating the derivation of ((9) from (20)), we obtain

$$W_i = \int_{M_i} \delta(\phi(v)) D\left(\frac{\phi}{v}\right) d^n v, \quad (28)$$

where  $D\left(\frac{\phi}{v}\right)$  is the usual Jacobian determinant of  $\vec{\phi}$  with respect to  $v$

$$\varepsilon^{A_1 \dots A_n} D\left(\frac{\phi}{v}\right) = \varepsilon^{\mu_1 \dots \mu_n} \partial_{\mu_1} m^{A_1} \partial_{\mu_2} m^{A_2} \dots \partial_{\mu_n} m^{A_n}. \quad (29)$$

According to the  $\delta$ -function theory[14], we know that  $\delta(\phi)$  can be expanded as

$$\delta(\phi) = \sum_{i=1}^l c_i \delta(N_i), \quad (30)$$

where the coefficients  $c_i$  must be positive, i.e.,  $c_i = |c_i|$ .  $\delta(N_i)$  is the  $\delta$ -function on a submanifold  $N_i$  in  $R^{2n-1}$  which had been given in Ref[13]

$$\delta(N_i) = \int_{N_i} \delta^{(2n-1)}(x - x_i(u)) \sqrt{g_u} d^{(n-1)}u, \quad (31)$$

where  $g_u = \det(g_{IJ})$  is the metric of submanifold  $N_i$ . Substituting Eq.(31) and Eq.(32) into Eq.(28), and calculating the integral, we get the expression of  $c_i$ ,

$$c_i = \frac{\beta_i}{\left| J\left(\frac{\phi}{v}\right) \right|_{p_i}} = \frac{\beta_i \eta_i}{\left| J\left(\frac{\phi}{v}\right) \right|_{p_i}}, \quad (32)$$

where the positive integer  $\beta_i = |W_i|$  is called the Hopf index of  $\phi$ -mapping on  $M_i$ , and  $\eta_i = \text{sgn}(J(\frac{\phi}{v}))|_{p_i} = \pm 1$  is the Brouwer degree[13]. So we find the relations between the Hopf index  $\beta_i$ , the Brouwer degree  $\eta_i$ , and the winding number  $W_i$

$$W_i = \beta_i \eta_i. \quad (33)$$

From Eq.(20), we have

$$\begin{aligned} j^{\mu_1 \mu_2 \cdots \mu_{n-1}} &= \delta^{(n)}(\phi) D^{\mu_1 \mu_2 \cdots \mu_{n-1}}\left(\frac{\phi}{x}\right) \\ &= \sum_{k=1}^l W_k \int_{N_k} \frac{D^{\mu_1 \mu_2 \cdots \mu_{n-1}}\left(\frac{\phi}{x}\right)}{D\left(\frac{\phi}{v}\right)}|_{N_i} \delta^{(2n-1)}(x - x_k(u)) \sqrt{g_u} d^{(n-1)}u \end{aligned} \quad (34)$$

Since on the singular submanifold  $N_i$  we have

$$\phi^a(x)|_{N_i} = \phi^a(x_i^1(u), \cdots, x_i^{(2n-1)}(u))|_{N_i} = 0, \quad (35)$$

which lead to

$$\partial_\mu \phi^a \frac{\partial x^\mu}{\partial u^I} = 0. \quad (36)$$

Using this expression, one can prove the relation between two Jacobian

$$D^{\mu_1 \mu_2 \cdots \mu_{n-1}}\left(\frac{\phi}{x}\right)|_{N_i} = \epsilon_{I_1 I_2 \cdots I_{n-1}} \frac{\partial x^{\mu_1}}{\partial u^{I_1}} \cdots \frac{\partial x^{\mu_{n-1}}}{\partial u^{I_{n-1}}} D\left(\frac{\phi}{v}\right). \quad (37)$$

Therefore, the inner topological structure of the topological current can be ultimately expressed as

$$j^{\mu_1 \mu_2 \cdots \mu_{n-1}} = \sum_k W_k \int_{N_k} \epsilon_{I_1 I_2 \cdots I_{n-1}} \frac{\partial x^{\mu_1}}{\partial u^{I_1}} \cdots \frac{\partial x^{\mu_{n-1}}}{\partial u^{I_{n-1}}} \delta^{(2n-1)}(x - x_k(u)) \sqrt{g_u} d^{(n-1)}u. \quad (38)$$



From this equation, we conclude that the inner structure of the topological current  $j^{\mu_1\mu_2\cdots\mu_{n-1}}$  is labelled by the total expansion of  $\delta(\phi)$ , which characters the  $l$  submanifolds  $N_k$  with the quantized topological charge  $W_k$ .

### III. HOPF INVARIANT

Now, we will discuss Hopf invariant in terms of the inner structure of the topological current obtained in the last section.

In terms of Eq.(9) Hopf invariant can be written as

$$H = \frac{1}{(n-1)!} \int \omega_{\mu_1\mu_2\cdots\mu_{n-1}} j^{\mu_1\mu_2\cdots\mu_{n-1}} d^{(2n-1)}x. \quad (39)$$

Substituting Eq.(37) into above equation, Hopf invariant is in the form

$$H = \frac{1}{(n-1)!} \sum_k W_k \int_{N_k} \epsilon_{I_1 I_2 \cdots I_{n-1}} \frac{\partial x^{\mu_1}}{\partial u^{I_1}} \cdots \frac{\partial x^{\mu_{n-1}}}{\partial u^{I_{n-1}}} \omega_{\mu_1\mu_2\cdots\mu_{n-1}} \sqrt{g_u} d^{(n-1)}u, \quad (40)$$

or the following form

$$H = \sum_k W_k \int_{N_k} \omega, \quad (41)$$

where  $N_k$  is a knotted(closed smooth oriented) submanifold of  $R^{2n-1}$ . Hopf invariant does not depend on the selection of the  $(n-1)$ -form  $\omega$ . It is difficult to continue the discussion directly from Eq.(40) or Eq.(41). But we find that it is useful to express  $\omega$  in terms of the topological current. For this purpose ,we introduce a new tensor

$$C_{\mu_1\mu_2\cdots\mu_{n-1}} = \epsilon_{\mu_1\mu_2\cdots\mu_{2n-1}} \partial_{\mu_n} j^{\mu_{n+1}\mu_{n+2}\cdots\mu_{2n-1}}. \quad (42)$$

If we impose the condition which is very similar with the selection of Columb gauge in classical electrodynamics

$$\partial_{\mu_i} \omega_{\mu_1\mu_2\cdots\mu_{n-1}} = 0, \quad i = 1, 2, \dots, n-1. \quad (43)$$

one can easily get

$$C_{\mu_1\mu_2\cdots\mu_{n-1}} = (n-1)! \partial_{\mu} \omega_{\mu\mu_1\mu_2\cdots\mu_{n-1}}. \quad (44)$$

Using the Green function formula

$$\Delta_{(2n-1)} \left( \frac{1}{\|x-y\|^{2n-3}} \right) = -(2n-3) A(S^{2n-2}) \delta^{(2n-1)}(x-y), \quad (45)$$

where  $\Delta_{(2n-1)} = \partial_\mu \partial_\mu$  is the  $(2n-1)$ -dimensional Laplacian operator in  $R^{2n-1}$ , the general formal solution of Eq.(44) is

$$\begin{aligned}\omega_{\mu_1 \mu_2 \dots \mu_{n-1}}(x) &= \frac{1}{(2n-3)A(S^{2n-2})(n-1)!} \int d^{(2n-1)}y \frac{C_{\mu_1 \mu_2 \dots \mu_{n-1}}(y)}{\|x-y\|^{2n-3}} \\ &= \frac{1}{(2n-3)A(S^{2n-2})(n-1)!} \epsilon_{\mu_1 \mu_2 \dots \mu_{2n-1}} \int d^{(2n-1)}y \frac{\partial_{\mu_n} j^{\mu_{n+1} \mu_{n+2} \dots \mu_{2n-1}}(y)}{\|x-y\|^{2n-3}}\end{aligned}\quad (46)$$

Because under the boundary condition (3) the topological current disappears on the boundary, one can easily get

$$\begin{aligned}\omega_{\mu_1 \mu_2 \dots \mu_{n-1}}(x) &= \frac{1}{(2n-3)A(S^{2n-2})(n-1)!} \\ &\quad \times \epsilon_{\mu_1 \mu_2 \dots \mu_{2n-1}} \int d^{(2n-1)}y j^{\mu_{n+1} \mu_{n+2} \dots \mu_{2n-1}}(y) \partial_{\mu_n} \frac{1}{\|x-y\|^{2n-3}}.\end{aligned}\quad (47)$$

Substituting it into Eq.(39), one get

$$\begin{aligned}H &= \frac{1}{(2n-3)A(S^{2n-2})[(n-1)!]^2} \epsilon_{\mu_1 \mu_2 \dots \mu_{2n-1}} \int d^{(2n-1)}x j^{\mu_1 \mu_2 \dots \mu_{n-1}}(x) \\ &\quad \times \int d^{(2n-1)}y j^{\mu_{n+1} \mu_{n+2} \dots \mu_{2n-1}}(y) \partial_{\mu_n} \frac{1}{\|x-y\|^{2n-3}}.\end{aligned}\quad (48)$$

According to the inner structure of topological current (38), one get

$$\begin{aligned}H &= \frac{1}{(2n-3)A(S^{2n-2})[(n-1)!]^2} \sum_{k,l} W_k W_l \epsilon_{\mu_1 \mu_2 \dots \mu_{2n-1}} \int_{N_k} \epsilon_{I_1 I_2 \dots I_{n-1}} \frac{\partial x^{\mu_1}}{\partial u^{I_1}} \dots \frac{\partial x^{\mu_{n-1}}}{\partial u^{I_{n-1}}} \sqrt{g_u} d^{(n-1)}u \\ &\quad \times \int_{N_l} \epsilon_{J_1 J_2 \dots J_{n-1}} \frac{\partial y^{\mu_{n+1}}}{\partial v^{J_1}} \dots \frac{\partial y^{\mu_{2n-1}}}{\partial v^{J_{n-1}}} \sqrt{g_v} d^{(n-1)}v \partial_{\mu_n} \frac{1}{\|x-y\|^{2n-3}}.\end{aligned}\quad (49)$$

Note that the integral is on the submanifolds  $N_k$  of  $R^{2n-1}$ , this gives the connection between Hopf invariant and the submanifolds  $N_k$ . To reveal the relationship between Hopf invariant and linking number of higher dimensional knots, we must seek a formula for the linking number of submanifolds of Euclidean space  $R^{2n-1}$ .

#### IV. LINKING NUMBER OF HIGHER DIMENSIONAL KNOT

Recall the definition of higher dimensional knot, in knot theory, a  $n$ -knot  $K$  is a knotted(closed, oriented)  $n$ -manifold. The strict definition of  $n$ -knot need the knowledge of Seifert surface. We are not intending to discuss the strict definition of higher dimensional knot. We are interested in two closed smooth oriented manifolds of the same dimensions  $(n-1)$  which is embedded in  $R^{2n-1}$  space[15]. Then the intersection number or linking number of

the two manifolds is well defined. In the classical case, knot, i.e. one dimensional knot  $\gamma$ , is in fact an embedding map to  $R^3$

$$\gamma : S^1 \rightarrow R^3. \quad (50)$$

The well known characteristic number for describes the topology of knots is linking number which is defined as the degree of map  $S^1 \times S^1 \rightarrow S^2$ . It is easy to generalize the ideas in the classical case into higher dimensional knot.

From Whitney's embedding theorem[6], we know that any smooth connected closed manifold  $M$  of dimension  $n$  can be smoothly embedded in  $R^{2n+1}$ . Consider two closed smooth oriented manifold  $N_k$  and  $N_l$  of the same dimensions  $(n-1)$ , the continuous mappings embedding  $N_k$  and  $N_l$  into  $R^{2n-1}$  that  $N_k$  and  $N_l$  do not intersect is

$$\begin{aligned} N_k : x^\mu &= x^\mu(u^1, u^2, \dots, u^{n-1}), \\ N_l : y^\mu &= y^\mu(v^1, v^2, \dots, v^{n-1}), \end{aligned} \quad (51)$$

where  $u^I, v^J (I, J = 1, 2, \dots, n-1)$  are the local parameters of  $N_k$  and  $N_l$  respectively. Consider the cartesian  $N_k \times N_l$  given the canonical orientation. We define a map

$$e : N_k \times N_l \rightarrow S^{2n-2}, \quad (52)$$

by associating to each  $(u, v) \in N_k \times N_l$  the unit vector is

$$e^\mu(u, v) = \frac{x^\mu(u) - y^\mu(v)}{\|x - y\|}, \quad \mu = 1, 2, \dots, 2n-1. \quad (53)$$

The degree of this map is the linking number  $Lk(N_k, N_l)$ . The volume element of the  $(2n-2)$  sphere  $S^{2n-2}$  is

$$d\sigma = \frac{1}{A(S^{2n-2})(2n-2)!} \epsilon_{\mu_1 \mu_2 \dots \mu_{2n-1}} e^{\mu_1} de^{\mu_2} \wedge \dots \wedge de^{\mu_{2n-1}}. \quad (54)$$

Then, clearly

$$\mathcal{L}k(N_k, N_l) = \deg e = \int_{N_k} \int_{N_l} e^* d\sigma. \quad (55)$$

The linking number  $\mathcal{L}k(N_k, N_l)$  is integer-valued and remains constant under the continuous varying of embedding mapping. Noticing that

$$de^\mu = \frac{dx^\mu - dy^\mu}{\|x - y\|} + (x^\mu - y^\mu) d \frac{1}{\|x - y\|}, \quad (56)$$

the volume element is

$$\begin{aligned}
d\sigma &= \frac{1}{A(S^{2n-2})(2n-2)!} \epsilon_{\mu_1 \mu_2 \dots \mu_{2n-1}} \frac{(x^{\mu_1} - y^{\mu_1})}{\|x - y\|^{2n-1}} (dx^{\mu_2} - dy^{\mu_2}) \wedge \dots \wedge (dx^{\mu_{2n-1}} - dy^{\mu_{2n-1}}) \\
&= \frac{1}{A(S^{2n-2})(2n-2)!} \epsilon_{\mu_1 \mu_2 \dots \mu_{2n-1}} \frac{(x^{\mu_1} - y^{\mu_1})}{\|x - y\|^{2n-1}} \left( \frac{\partial x^{\mu_2}}{\partial u^{I^2}} du^{I^2} - \frac{\partial y^{\mu_2}}{\partial v^{J^2}} dv^{J^2} \right) \wedge \dots \\
&\quad \wedge \left( \frac{\partial x^{\mu_{2n-1}}}{\partial u^{I^{2n-1}}} du^{I^{2n-1}} - \frac{\partial y^{\mu_{2n-1}}}{\partial v^{J^{2n-1}}} dv^{J^{2n-1}} \right).
\end{aligned} \tag{57}$$

The linking number  $Lk(N_k, N_l)$  is ultimately expressed as

$$\begin{aligned}
\mathcal{L}k(N_k, N_l) &= \frac{1}{(2n-3)A(S^{2n-2})[(n-1)!]^2} \epsilon_{\mu_1 \mu_2 \dots \mu_{2n-1}} \int_{N_k} \epsilon_{I_1 I_2 \dots I_{n-1}} \frac{\partial x^{\mu_1}}{\partial u^{I_1}} \dots \frac{\partial x^{\mu_{n-1}}}{\partial u^{I_{n-1}}} \sqrt{g_u} d^{(n-1)}u \\
&\quad \times \int_{N_l} \epsilon_{J_1 J_2 \dots J_{n-1}} \frac{\partial y^{\mu_{n+1}}}{\partial v^{J_1}} \dots \frac{\partial y^{\mu_{2n-1}}}{\partial v^{J_{n-1}}} \sqrt{g_v} d^{(n-1)}v \frac{1}{\partial_{\mu_n} \|x - y\|^{2n-3}}.
\end{aligned} \tag{58}$$

At last, comparing the above expression with Eq.(48) one get Hopf invariant

$$H = \sum_{k,l} W_k W_l \mathcal{L}k(N_k, N_l). \tag{59}$$

This expression for Hopf invariant is consistent with the interpretation of Hopf invariant from mathematical literature[8] which is the linking number of the preimages  $N_k = f^{-1}(p)$  and  $N_l = f^{-1}(q)$  of any two different values of Hopf mapping  $f$ .

At last, to complete this paper, a final remark is necessary. Following Alberto S. Cattaneo's work on **BF** topological field theory[16], similarly, we can construct a generalized abelian field theory of Hopf invariant's type which is defined by the action functional

$$S(A) = \int_M A \wedge dA, \tag{60}$$

where  $A$  is an ordinary  $(n-1)$ -form on an  $(2n-1)$ -dimensional manifold  $M$  with  $n$  being even. The basic field or independent field variable is the  $(n-1)$  order tensor field  $A_{\mu_1 \dots \mu_{n-1}}$  which probably describes the high spin particles. The generalization of abelian gauge symmetries is in this case given by transformations of the form  $A \mapsto A + d\sigma, \sigma \in \Omega^{n-2}(M)$ . As pointed out in this paper, the explanation of the action (60) is an topological invariant which in the case  $M = R^{2n-1}$  turns out to be a function of the linking number of the higher-dimensional knot. The classical Euler-Lagrange equations of motion for the action gives an identical equation. As in Chern-Simons theory, field variable  $A_{\mu_1 \dots \mu_{n-1}}$  should be coupled with the matter field or the matter current to display the property of the action. In quantum field theory, in addition to the action functional, one also wishes to pick a suitable class of gauge

invariant observables. As done by E.Witten in his pioneer work[17] which gives a theoretical framework to study the relationship between Chern-Simons action and the knot invariant, the obvious generalization of a Wilson loop as a observables has the form

$$W(N_i) = \exp\{\frac{i}{\hbar}\lambda \int_{N_i} A\}, \quad (61)$$

where  $N_i$  is a knotted(closed oriented)  $(n-1)$ -dimensional submanifolds of  $M$  and  $\lambda$  is the coupling constant. Then we propose the Feynman path integral

$$\int \mathcal{D}A \exp\{\frac{i}{\hbar}S\} \prod_{i=1}^l W(N_i). \quad (62)$$

The generalization of Wilson loops where the connection is replaced by a form  $A$  of higher degree and the loop by a higher-dimensional submanifold is then natural and might have applications to the theories of  $\tilde{p}$ -brane. In terms of the  $\phi$ -mapping topological current theory,  $(n-1)$  dimensional submanifold can be naturally deduced from the action (60), which plays a crucial role in realization of the  $\tilde{p}$ -brane scenario. If the action (60) is significative, a further study of the generalized abelian field theory and the application to  $\tilde{p}$ -brane will attract our future attention.

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